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Distance Coordinates With Respect to a Triangle of Reference

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With respect to a triangle of reference $A_1A_2A_3$, each point P in the plane of the triangle, has unique *area* coordinates: $P = (b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 1$. *Distance* coordinates are introduced such that $P=[d_1, d_2, d_3]$, with d_k the distance from P to A_k. It is shown that there is an explicit function $f(x_1, x_2, x_3)$ such that $f(d_1^2, d_2^2, d_3^2) = 0$ is necessary and sufficient for $P = [d_1, d_2, d_3]$, each d_k nonnegative. The partial derivatives $f_k(x_1, x_2, x_3) = \partial f(x_1, x_2, x_3)/\partial x_k$ are such that $b_k = f_k(d_1^2, d_2^2, d_3^2)$ for each k. Other results relating the b_k and the d_k are given. The use of $f(x_1, x_2, x_3)$ in solving geometric problems is shown.

Key words: Area coordinates; distance coordinates; Plane Geometry; radical center; triangle of reference.

We are given three noncollinear points A_1, A_2, A_3 and all other points are in the plane of the triangle of reference $A_1A_2A_3$.

Notationally, two distinct points X, Y determine an infinite line XY, with the finite line segment $\overline{XY}$ having length $|X, Y|$. If X, Y are centers of circles with radii x, y respectively, the *radical axis* of those circles is a line perpendicular to XY, at a point which is $(|X, Y|^2 + x^2 - y^2)/2|X, Y|$ from X in the direction of Y. Given a third point Z, not on XY, as the center of a circle, the three radical axes meet in a common point called their *radical center*. The area of the triangle XYZ is denoted by $|X, Y, Z|$. The function

(1) $$F(x_1, x_2, x_3) = 2(x_1x_2 + x_1x_3 + x_2x_3) - x_1^2 - x_2^2 - x_3^2$$

has the well-known property

(2) $$16|X, Y, Z|^2 = F(|X, Y|^2, |X, Z|^2, |Y, Z|^2).$$

Let $\Delta = |A_1, A_2, A_3|$ denote the area of the triangle of reference. With respect to this triangle every point P has unique *area coordinates*[1] b_1, b_2, b_3, which are real numbers restricted by

(3) $$b_1 + b_2 + b_3 = 1.$$

We write

$$P = (b_1, b_2, b_3),$$

with $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 1)$.

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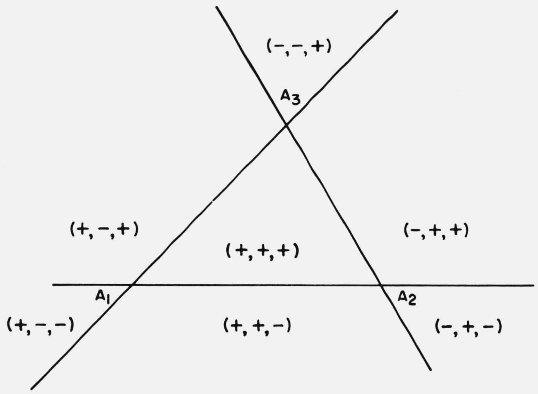
[1] Also called "normalized barycentric" or "areal" coordinates [1].[2]

[2] Coxeter, H.S.M., Introduction to Geometry (Wiley, 1961).

The area coordinate b_1 is defined as the ratio $\pm|P, A_2, A_3|/\Delta$ with $b_1 > 0$ if $\overline{A_1P}$ does not intersect A_2A_3, $b_1 \leqslant 0$ if it does. Similarly for the other b_k, so that the diagram of signs is



Conversely, any three real numbers b_1, b_2, b_3, restricted by (3), define a unique point $P = (b_1, b_2, b_3)$ with respect to $A_1A_2A_3$.

If a point P is at a distance d_1 from A_1, d_2 from A_2, and d_3 from A_3, we call these the *distance coordinates* of P with respect to $A_1A_2A_3$, and write

$$P = [d_1, d_2, d_3],$$

or, when more convenient,

$$P = < d_1^2, d_2^2, d_3^2 >.$$

For clarity, $d_k = |P, A_k|$ *for all* $k = 1, 2, 3$.

Note that the triangle PA_2A_3 has side lengths a_1, d_2, d_3 so that

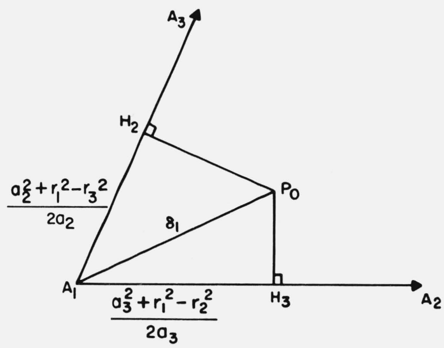
(4) $$16\Delta^2 b_1^2 = F(a_1^2, d_2^2, d_3^2)$$

with similar equations for the other b_k. Also note that P is the radical center of circles with centers A_1, A_2, A_3 and radii d_1, d_2, d_3 respectively.

Now consider the general case of circles with centers A_1, A_2, A_3 and radii r_1, r_2, r_3. Denote their radical center by

(5) $$P_0 = [\delta_1, \delta_2, \delta_3].$$

In what follows there is no loss of generality in assuming that P_0 is in the interior of $A_1A_2A_3$. Near A_1 we have



Let α_1 denote the interior angle at A_1. Then $|H_2, H_3| = \delta_1 \sin \alpha_1$.[3] The formula for $\cos \alpha_1$ in the triangle $A_1H_2H_3$ is

$$(6)\qquad 2\left(\frac{a_2^2+r_1^2-r_3^2}{2a_2}\right)\left(\frac{a_3^2+r_1^2-r_2^2}{2a_3}\right)\cos\alpha_1=\left(\frac{a_2^2+r_1^2-r_3^2}{2a_2}\right)^2+\left(\frac{a_3^2+r_1^2-r_2^2}{2a_3}\right)^2-(\delta_1\sin\alpha_1)^2.$$

In order to simplify this equation we shall need a few definitions and formulas. Define

$$(7)\qquad \begin{aligned} c_1 &= a_2^2+a_3^2-a_1^2\\ c_2 &= a_1^2+a_3^2-a_2^2\\ c_3 &= a_1^2+a_2^2-a_3^2. \end{aligned}$$

Note that

$$(8)\qquad c_1 = 2a_2^2 - c_3 = 2a_3^2 - c_2$$

and that

$$(9)\qquad 2a_2a_3\cos\alpha_1 = c_1.$$

Since

$$(10)\qquad a_2a_3\sin\alpha_1 = 2\Delta,$$

we have

$$(11)\qquad c_1^2 + 16\Delta^2 = 4a_2^2a_3^2.$$

Now return to eq (6). Multiply through by $4a_2^2a_3^2$, and use eqs (7) through (11) to simplify. We get $16\Delta^2 f(r_1^2, r_2^2, r_3^2) = 16\Delta^2(r_1^2-\delta_1^2)$, where

$$(12)\qquad 16\Delta^2 f(x_1,x_2,x_3) = \sum_{k=1}^{3} a_k^2c_kx_k - a_1^2a_2^2a_3^2 - \frac{1}{2}\{c_1(x_2-x_3)^2+c_2(x_1-x_3)^2+c_3(x_1-x_2)^2\}.$$

Generally:

$$(13)\qquad f(r_1^2, r_2^2, r_3^2) = r_k^2 - \delta_k^2. \qquad k = 1, 2, 3.$$

If $f(r_1^2, r_2^2, r_3^2) = 0$ then $r_k = \delta_k$ for all $k = 1, 2, 3$ so that $P_0 = [r_1, r_2, r_3]$.
Now notice that

$$(14)\qquad \sum_{k=1}^{3} a_k^2c_k = F(a_1^2, a_2^2, a_3^2) = 16\Delta^2,$$

[3] The circle with diameter $\overline{A_1P_0}$ goes through H_2 and H_3. Thus $\overline{H_2H_3}$ is a chord with opposite angle α_1.

so that

$$f(x_1 - t, x_2 - t, x_3 - t) = f(x_1, x_2, x_3) - t \qquad \text{all } t. \tag{15}$$

Let $x_k = r_k^2$ for all $k = 1, 2, 3$ and let $t = f(r_1^2, r_2^2, r_3^2)$. Since (13) implies $x_k - t = \delta_k^2$ for all $k = 1, 2, 3$, we have $f(\delta_1^2, \delta_2^2, \delta_3^2) = 0$. As we pointed out before, any point $P = [d_1, d_2, d_3]$ can be a radical center. The above shows that $f(d_1^2, d_2^2, d_3^2) = 0$.

Combining this with our remarks following eq (13), we have proved the first part of the following theorem:

THEOREM. *Let* d_1, d_2, d_3 *be any real nonnegative numbers. Then there is a point* $P = [d_1, d_2, d_3]$ *if and only if* $f(d_1^2, d_2^2, d_3^2) = 0$ (f *defined in* (12)). *In that case* $P = (f_1(d_1^2, d_2^2, d_3^2), f_2(d_1^2, d_2^2, d_3^2), f_3(d_1^2, d_2^2, d_3^2))$ *where* $f_k(x_1, x_2, x_3) = \partial f(x_1, x_2, x_3)/\partial x_k$ *for all* $k = 1, 2, 3$.

For clarity, we write out the f_k:

$$\begin{aligned} 16\Delta^2 f_1(x_1, x_2, x_3) &= a_1^2 c_1 + c_3(x_2 - x_1) + c_2(x_3 - x_1) \\ 16\Delta^2 f_2(x_1, x_2, x_3) &= a_2^2 c_2 + c_3(x_1 - x_2) + c_1(x_3 - x_2) \\ 16\Delta^2 f_3(x_1, x_2, x_3) &= a_3^2 c_3 + c_2(x_1 - x_3) + c_1(x_2 - x_3). \end{aligned} \tag{16}$$

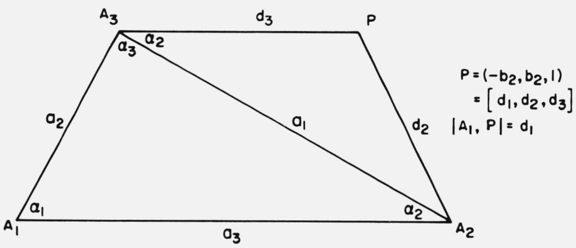
Note that (14) implies

$$f_1(x_1, x_2, x_3) + f_2(x_1, x_2, x_3) + f_3(x_1, x_2, x_3) = 1. \tag{17}$$

Also note that

$$f_k(x_1 - t, x_2 - t, x_3 - t) = f_k(x_1, x_2, x_3) \qquad \text{all } t;\ k = 1, 2, 3. \tag{18}$$

In the process of proving the last part of the theorem, we shall need the following computations. The locus of points with $b_3 = 1$ is a line through A_3 parallel to A_1A_2. Naturally if $P = (b_1, b_2, b_3)$ then $b_1 + b_2 = 0$ (see (3)). Suppose $b_1 \leqslant 0$.



The area of PA_2A_3 is $\Delta \cdot |-b_2|$. It is also $\frac{1}{2} a_1 d_3 \sin \alpha_2$.

Therefore

$$d_3 = a_3 b_2. \tag{19}$$

Using the cosine formula in that triangle yields

$$\begin{aligned} d_2^2 &= d_3^2 + a_1^2 - 2a_1 d_3 \cos \alpha_2 \\ &= a_3^2 b_2^2 + a_2^2 - b_2 c_2. \end{aligned} \tag{20}$$

Using the cosine formula in the triangle A_1A_3P we have

$$d_1^2 = d_3^2 + a_2^2 - 2a_2 d_3 \cos(\alpha_2 + \alpha_3)$$

(21) $$= a_3^2 b_2^2 + a_2^2 + b_2 c_1.$$

Now use eq (18) with each $x_k = d_k^2$ and $t = d_3^2 = a_3^2 b_2^2$. For $k=1$ we have (using (20) and (21)):

$$16\Delta^2 f_1(d_1^2, d_2^2, d_3^2) = 16\Delta^2 f_1(a_2^2 + b_2 c_1, a_1^2 - b_2 c_2, 0)$$

$$= a_1^2 c_1 + c_3(a_1^2 - a_2^2 - b_2(c_1 + c_2)) - c_2(a_2^2 + b_2 c_1)$$

(using (16)). Since $a_1^2 c_1 + c_3(a_1^2 - a_2^2) = c_2 a_2^2$ (note $c_1 + c_3 = 2a_2^2$, $c_2 + c_3 = 2a_1^2$), and since

(22) $$c_1 c_2 + c_1 c_3 + c_2 c_3 = 16\Delta^2,$$

($c_1 + c_2 = 2a_3^2$, then add the three symmetric formulas, and use eq (14)), we have

$$f_1(d_1^2, d_2^2, d_3^2) = -b_2 = b_1$$

as desired.

Similar computations prove $f_2(d_1^2, d_2^2, d_3^2) = b_2$ and (as a check), $f_3\ (d_1^2, d_2^2, d_3^2) = 1 = b_3$. The case $b_2 \leqslant 0$ is handled similarly to prove the last part of the theorem for the case $b_3 = 1$.

Return to eq (12) and solve $f(x_1, x_2, x_3) = 0$ for x_1. An intermediary stage is the equation

(23) $$16\Delta^2\{f_1(x_1, x_2, x_3)\}^2 = 2a_1^2(x_2 + x_3) + 2x_2 x_3 - a_1^4 - x_2^2 - x_3^2.$$

The r.h.s. is recognized to be $F(a_1^2, x_2, x_3)$. Let $x_k = d_k^2$ for all $k = 1,2,3$, and use (4), to get $\{f_1(d_1^2, d_2^2, d_3^2)\}^2 = b_1^2$. Generally

$$f_k(d_1^2, d_2^2, d_3^2) = \pm\, b_k \qquad k = 1,2,3$$

Set $f_k = f_k(d_1^2, d_2^2, d_3^2)$ for all $k = 1, 2, 3$. Equations (3) and (17), showing $\Sigma b_k = \Sigma f_k = 1$, imply that we cannot have $f_k = -b_k$ for all k. Suppose $f_3 = b_3$. Then

$$f_1 + f_2 = 1 - f_3 = 1 - b_3 = b_1 + b_2.$$

If $f_1 = b_1$ then $f_2 = b_2$, and conversely. The only open case is $f_1 = -b_1$, $f_2 = -b_2$. This implies $b_1 + b_2 = 0$, whence $b_3 = 1$. We have already covered this case, so the proof of the theorem is complete.

An interesting implication for P_0 is immediate. Use eq (18) with $x_k = r_k^2$ for all $k = 1, 2, 3$ and $t = f(r_1^2, r_2^2, r_3^2)$ as before. The result is $f_k(\delta_1^2, \delta_2^2, \delta_3^2) = f_k(r_1^2, r_2^2, r_3^2)$ for all $k = 1, 2, 3$. In other words the area coordinates for the radical center of three circles with centers at A_1, A_2, A_3 and radii r_1, r_2, r_3 respectively are given by

(24) $$P_0 = (f_1(r_1^2, r_2^2, r_3^2), f_2(r_1^2, r_2^2, r_3^2), f_3(r_1^2, r_2^2, r_3^2)).$$

Of course, the distance coordinates are given by

(25) $$P_0 = < r_1^2 - f(r_1^2, r_2^2, r_3^2), r_2^2 - f(r_1^2, r_2^2, r_3^2), r_3^2 - f(r_1^2, r_2^2, r_3^2) >.$$

If we wish to find the (0 to 8) circles simultaneously tangent to the three circles used above, we can do so through f, to obtain four quadratic equations whose solutions solve the problem. The point is that a circle of radius r which is simultaneously tangent to all three circles has a center

(26) $$P=[r_1+\epsilon_1 r,\ r_2+\epsilon_2 r,\ r_3+\epsilon_3 r],$$

with each $\epsilon_k=\pm 1$ depending on whether the tangency is "outside" ($\epsilon=1$) or "inside" ($\epsilon=-1$). Simplifying

(27) $$f((r_1+\epsilon_1 r)^2,\ (r_2+\epsilon_2 r)^2,\ (r_3+\epsilon_3 r)^2)=0,$$

we get a quadratic equation in r (with constant term $f(r_1^2, r_2^2, r_3^2)$). If r is a negative root of that equation, we simply "assign" $-r$ to $-\epsilon_1, -\epsilon_2, -\epsilon_3$ since $\epsilon_k r=(-\epsilon_k)(-r)$, $k=1, 2, 3$.

Thus we can cover all solutions with just four triples of ϵ's, no two of which are negatives of each other.

We end this note with a list of formulas connecting the area and distance coordinates of a point

$$P=(b_1, b_2, b_3)=[d_1, d_2, d_3].$$

The formulas are given without proof, but are easily derived, with extensive use of the formula for the distance between P and $P'=(b_1', b_2', b_3')$:

(28) $$2\,|P, P'|^2=\sum_{k=1}^{3} c_k(b_k-b_k')^2.$$

First we complete the connection between the coordinates, begun in the last formula of the theorem, with

(29) $$\begin{aligned} d_1^2&=a_3^2b_2^2+c_1b_2b_3+a_2^2b_3^2\\ d_2^2&=a_3^2b_1^2+c_2b_1b_3+a_1^2b_3^2\\ d_3^2&=a_2^2b_1^2+c_3b_1b_2+a_1^2b_2^2 \end{aligned}$$

or

(30) $$2d_k^2=(1-2b_k)c_k+\sum_{n=1}^{3} b_n^2c_n \qquad k=1,2,3\cdot$$

Let R denote the circumradius of $A_1A_2A_3$, and ρ_P the distance from P to the circumcenter. The latter has area coordinates $a_k^2c_k/16\Delta^2$, $k=1,2,3$. Also $4\Delta R=a_1a_2a_3$.

Define

(31) $$G_P=R^2-\rho_P^2\cdot$$

Then G_P can be found using only the b_k:

(32) $$G_P=a_1^2b_2b_3+a_2^2b_1b_3+a_3^2b_1b_2$$

(33) $$=\frac{1}{2}\sum_{k=1}^{3}(b_k-b_k^2)c_k;$$

or only the d_k:

(34) $$G_P=R^2-(c_1(d_2-d_3)^2+c_2(d_1-d_3)^2+c_3(d_1-d_2)^2)/32\Delta^2$$

$$(35)\qquad = R^2 - (a_1^2(d_1^2-d_2^2)(d_1^2-d_3^2) + a_2^2(d_2^2-d_1^2)(d_2^2-d_3^2) + a_3^2(d_3^2-d_1^2)(d_3^2-d_2^2))/16\Delta^2$$

$$(36)\qquad = (2a_1^2a_2^2a_3^2 - \sum a_k^2c_kd_k^2)/16\Delta^2;$$

or symmetric combinations:

$$(37)\qquad G_P = \sum_{k=1}^{3} b_kd_k^2$$

$$(38)\qquad = \sum_{k=1}^{3} (a_k^2 - 2d_k^2 + b_k^2c_k)/4$$

$$(39)\qquad = \sum_{k=1}^{3} (a_k^2 - 2d_k^2 + b_kc_k)/6$$

$$(40)\qquad = \sum_{k=1}^{3} (a_k^2 - d_k^2 - b_ka_k^2)/3$$

$$(41)\qquad = 2\left\{b_1a_2^2a_3^2 + b_2a_1^2a_3^2 + b_3a_1^2a_2^2 - \sum_{k=1}^{3} a_k^2d_k^2\right\}\Big/\sum_{k=1}^{3} a_k^2$$

$$(42)\qquad = \left\{b_1b_2b_3\sum_{k=1}^{3} a_k^2 + \sum_{k=1}^{3} b_k^2d_k^2\right\}\Big/2(b_1b_2 + b_1b_3 + b_2b_3)$$

(except at a vertex). Other relations are

$$(43)\qquad d_1^2 = b_2a_3^2 + b_3a_2^2 - G_P \qquad \text{etc.}$$

$$(44)\qquad b_1c_1 = 2G_P - a_1^2 + d_2^2 + d_3^2 \qquad \text{etc.}$$

$$(45)\qquad (1-b_1)G_P = b_1d_1^2 + b_2b_3a_1^2 \qquad \text{etc.}$$

$$(46)\qquad 2a_2^2a_3^2b_1 = c_1G_P + a_2^2d_2^2 + a_3^2d_3^2 - a_1^2d_1^2 \qquad \text{etc.}$$

The pedal triangle of P, which has side lengths $a_kd_k/2R$, $k=1, 2, 3$, *has area* $\Delta|G_P|/4R^2$; i.e.,

$$(47)\qquad 16\Delta^2G_P^2 = F(a_1^2d_1^2, a_2^2d_2^2, a_3^2d_3^2)$$

$$(48)\qquad = 4a_2^2a_3^2d_2^2d_3^2 - (a_2^2d_2^2 + a_3^2d_3^2 - a_1^2d_1^2)^2 \qquad \text{etc.}$$

If P does not lie on the triangle $A_1A_2A_3$ then the lines A_1P, A_2P, A_3P intersect the circle of radius ρ_P, concentric with the circumcircle, in P and in points A_1', A_2', A_3' respectively which have (opposite) side lengths $\lambda_P|b_k|d_k$ for $k=1, 2, 3$; $\lambda_P = 4\Delta\rho_P/d_1d_2d_3$. The area of $A_1'A_2'A_3'$ is $\Delta\lambda_P^2|b_1b_2b_3|$. Thus

$$(49)\qquad 16\Delta^2(b_1b_2b_3)^2 = F(b_1^2d_1^2, b_2^2d_2^2, b_3^2d_3^2).$$

Finally, suppose $d_k^2 = g_k(t)$, a differentiable function of t, for $k=1, 2, 3$. The b_k will also be differentiable functions of t, (by the last statement of the theorem), and b_k' will denote the derivative. We have

$$\sum_{k=1}^{3} b_k g_k'(t) = 0 \tag{50}$$

and

$$g_1'(t) + c_1 b_1' = g_2'(t) + c_2 b_2' = g_3'(t) + c_3 b_3' = \sum_{k=1}^{3} c_k b_k b_k'. \tag{51}$$

(Paper 76B3&4–367)