Gauss' Law

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Chapter 1

Gauss's law

This article is about Gauss's law concerning the electric field. For analogous law concerning different fields, see Gauss's law for magnetism and Gauss's law for gravity. For Gauss's theorem, a mathematical theorem relevant to all of these laws, see Divergence theorem.

In physics, **Gauss's law**, also known as **Gauss's flux theorem**, is a law relating the distribution of electric charge to the resulting electric field.

The law was formulated by Carl Friedrich Gauss in 1835, but was not published until 1867.[1] It is one of Maxwell's four equations, which form the basis of classical electrodynamics, the other three being Gauss's law for magnetism, Faraday's law of induction, and Ampère's law with Maxwell's correction. Gauss's law can be used to derive Coulomb's law,[2] and vice versa.

1.1 Qualitative description

In words, Gauss's law states that:

> *The net electric flux through any closed surface is equal to* $^1\!/\!_{\varepsilon}$ *times the net electric charge enclosed within that closed surface.*[3]

Gauss's law has a close mathematical similarity with a number of laws in other areas of physics, such as Gauss's law for magnetism and Gauss's law for gravity. In fact, any "inverse-square law" can be formulated in a way similar to Gauss's law: For example, Gauss's law itself is essentially equivalent to the inverse-square Coulomb's law, and Gauss's law for gravity is essentially equivalent to the inverse-square Newton's law of gravity.

Gauss's law is something of an electrical analogue of Ampère's law, which deals with magnetism.

The law can be expressed mathematically using vector calculus in integral form and differential form, both are equivalent since they are related by the divergence theorem, also called Gauss's theorem. Each of these forms in turn can also be expressed two ways: In terms of a relation between the electric field **E** and the total electric charge, or in terms of the electric displacement field **D** and the *free* electric charge.[4]

1.2 Equation involving E field

Gauss's law can be stated using either the electric field **E** or the electric displacement field **D**. This section shows some of the forms with **E**; the form with **D** is below, as are other forms with **E**.

1.2.1 Integral form

Gauss's law may be expressed as:[5]

$$\Phi_E = \frac{Q}{\varepsilon_0}$$

where ΦE is the electric flux through a closed surface S enclosing any volume V, Q is the total charge enclosed within S, and ε_0 is the electric constant. The electric flux ΦE is defined as a surface integral of the electric field:

$$\Phi_E = \oiint\ {}_S\, \mathbf{E} \cdot \mathrm{d}\mathbf{A}$$

where **E** is the electric field, d**A** is a vector representing an infinitesimal element of area,[note 1] and · represents the dot product of two vectors.

Since the flux is defined as an *integral* of the electric field, this expression of Gauss's law is called the *integral form*.

Applying the integral form

Main article: Gaussian surface
See also Capacitance (Gauss's law)

If the electric field is known everywhere, Gauss's law makes it quite easy, in principle, to find the distribution of electric charge: The charge in any given region can be deduced by integrating the electric field to find the flux.

However, much more often, it is the reverse problem that needs to be solved: The electric charge distribution is known, and the electric field needs to be computed. This is much more difficult, since if you know the total flux through a given surface, that gives almost no information about the electric field, which (for all you know) could go in and out of the surface in arbitrarily complicated patterns.

An exception is if there is some symmetry in the situation, which mandates that the electric field passes through the surface in a uniform way. Then, if the total flux is known, the field itself can be deduced at every point. Common examples of symmetries which lend themselves to Gauss's law include cylindrical symmetry, planar symmetry, and spherical symmetry. See the article Gaussian surface for examples where these symmetries are exploited to compute electric fields.

1.2.2 Differential form

By the divergence theorem, Gauss's law can alternatively be written in the *differential form*:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

where $\nabla \cdot \mathbf{E}$ is the divergence of the electric field, ε_0 is the electric constant, and ϱ is the total electric charge density (charge per unit volume).

1.2.3 Equivalence of integral and differential forms

Main article: Divergence theorem

The integral and differential forms are mathematically equivalent, by the divergence theorem. Here is the argument more specifically.

1.3 Equation involving D field

See also: Maxwell's equations

1.3.1 Free, bound, and total charge

Main article: Electric polarization

The electric charge that arises in the simplest textbook situations would be classified as "free charge"—for example, the charge which is transferred in static electricity, or the charge on a capacitor plate. In contrast, "bound charge" arises only in the context of dielectric (polarizable) materials. (All materials are polarizable to some extent.) When such materials are placed in an external electric field, the electrons remain bound to their respective atoms, but shift a microscopic distance in response to the field, so that they're more on one side of the atom than the other. All these microscopic displacements add up to give a macroscopic net charge distribution, and this constitutes the "bound charge".

Although microscopically, all charge is fundamentally the same, there are often practical reasons for wanting to treat bound charge differently from free charge. The result is that the more "fundamental" Gauss's law, in terms of **E** (above), is sometimes put into the equivalent form below, which is in terms of **D** and the free charge only.

1.3.2 Integral form

This formulation of Gauss's law states the total charge form:

$$\Phi_D = Q_{\text{free}}$$

where ΦD is the **D**-field flux through a surface S which encloses a volume V, and Q_{free} is the free charge contained in V. The flux ΦD is defined analogously to the flux ΦE of the electric field **E** through S:

$$\Phi_D = \oiint_S \mathbf{D} \cdot \mathrm{d}\mathbf{A}$$

1.3.3 Differential form

The differential form of Gauss's law, involving free charge only, states:

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}}$$

where $\nabla \cdot \mathbf{D}$ is the divergence of the electric displacement field, and ϱ_{free} is the free electric charge density.

1.4 Equivalence of total and free charge statements

1.5 Equation for linear materials

In homogeneous, isotropic, nondispersive, linear materials, there is a simple relationship between **E** and **D**:

$\mathbf{D} = \varepsilon\mathbf{E}$

where ε is the permittivity of the material. For the case of vacuum (aka free space), $\varepsilon = \varepsilon_0$. Under these circumstances, Gauss's law modifies to

$$\Phi_E = \frac{Q_{\text{free}}}{\varepsilon}$$

for the integral form, and

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{free}}}{\varepsilon}$$

for the differential form.

1.6 Relation to Coulomb's law

1.6.1 Deriving Gauss's law from Coulomb's law

Gauss's law can be derived from Coulomb's law.

Note that since Coulomb's law only applies to *stationary* charges, there is no reason to expect Gauss's law to hold for moving charges *based on this derivation alone*. In fact, Gauss's law *does* hold for moving charges, and in this respect Gauss's law is more general than Coulomb's law.

1.6.2 Deriving Coulomb's law from Gauss's law

Strictly speaking, Coulomb's law cannot be derived from Gauss's law alone, since Gauss's law does not give any information regarding the curl of $\mathbf{E}$ (see Helmholtz decomposition and Faraday's law). However, Coulomb's law *can* be proven from Gauss's law if it is assumed, in addition, that the electric field from a point charge is spherically-symmetric (this assumption, like Coulomb's law itself, is exactly true if the charge is stationary, and approximately true if the charge is in motion).

1.7 See also

- Method of image charges
- Uniqueness theorem for Poisson's equation

1.8 Notes

[1] More specifically, the infinitesimal area is thought of as planar and with area dA. The vector d$\mathbf{A}$ is normal to this area element and has magnitude dA.[6]

1.9 References

[1] Bellone, Enrico (1980). *A World on Paper: Studies on the Second Scientific Revolution*.

[2] Halliday, David; Resnick, Robert (1970). *Fundamentals of Physics*. John Wiley & Sons, Inc. pp. 452–53.

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[4] I.S. Grant, W.R. Phillips (2008). *Electromagnetism* (2nd ed.). Manchester Physics, John Wiley & Sons. ISBN 978-0-471-92712-9.

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[6] Matthews, Paul (1998). *Vector Calculus*. Springer. ISBN 3-540-76180-2.

[7] See, for example, Griffiths, David J. (2013). *Introduction to Electrodynamics (4th ed.)*. Prentice Hall. p. 50.

Jackson, John David (1998). Classical Electrodynamics, 3rd ed., New York: Wiley. ISBN 0-471-30932-X.

1.10 External links

- MIT Video Lecture Series (30 x 50 minute lectures)- Electricity and Magnetism Taught by Professor Walter Lewin.
- section on Gauss's law in an online textbook
- MISN-0-132 *Gauss's Law for Spherical Symmetry* (PDF file) by Peter Signell for Project PHYSNET.
- MISN-0-133 *Gauss's Law Applied to Cylindrical and Planar Charge Distributions* (PDF file) by Peter Signell for Project PHYSNET.

Chapter 2

Electric flux

In electromagnetism, **electric flux** is the measure of flow of the electric field through a given area. Electric flux is proportional to the number of electric field lines going through a normally perpendicular surface. If the electric field is uniform, the electric flux passing through a surface of vector area **S** is

$$\Phi_E = \mathbf{E} \cdot \mathbf{S} = ES\cos\theta,$$

where **E** is the electric field (having units of V/m), E is its magnitude, S is the area of the surface, and θ is the angle between the electric field lines and the normal (perpendicular) to S.

For a non-uniform electric field, the electric flux $d\Phi E$ through a small surface area $d\mathbf{S}$ is given by

$$d\Phi_E = \mathbf{E} \cdot d\mathbf{S}$$

(the electric field, **E**, multiplied by the component of area perpendicular to the field). The electric flux over a surface S is therefore given by the surface integral:

$$\Phi_E = \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where **E** is the electric field and $d\mathbf{S}$ is a differential area on the closed surface S with an outward facing surface normal defining its direction.

For a closed Gaussian surface, electric flux is given by:

$$\Phi_E = \oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

where

E is the electric field,

S is any closed surface,

Q is the total electric charge inside the surface S,

ε_0 is the electric constant (a universal constant, also called the "permittivity of free space”) ($\varepsilon_0 \approx 8.854\ 187\ 817... \times 10^{-12}$ farads per meter ($F \cdot m^{-1}$)).

This relation is known as Gauss’ law for electric field in its integral form and it is one of the four Maxwell’s equations.

While the electric flux is not affected by charges that are not within the closed surface, the net electric field, **E**, in the Gauss’ Law equation, can be affected by charges that lie outside the closed surface. While Gauss’ Law holds for all situations, it is only useful for “by hand” calculations when high degrees of symmetry exist in the electric field. Examples include spherical and cylindrical symmetry.

Electrical flux has SI units of volt metres ($V\,m$), or, equivalently, newton metres squared per coulomb ($N\,m^2\,C^{-1}$). Thus, the SI base units of electric flux are $kg{\cdot}m^3{\cdot}s^{-3}{\cdot}A^{-1}$.

Its dimensional formula is $[L^3MT^{-3}I^{-1}]$.

2.1 See also

- Magnetic flux
- Maxwell’s equations

related websites are following: http://www.citycollegiate.com/coulomb4_XII.htm[1]

2.2 References

[1] electric flux

2.3 External links

- Electric flux — HyperPhysics

Chapter 3

Ampère's circuital law

"Ampère's law" redirects here. For the law describing forces between current-carrying wires, see Ampère's force law.

In classical electromagnetism, **Ampère's circuital law**, discovered by André-Marie Ampère in 1826,[1] relates the integrated magnetic field around a closed loop to the electric current passing through the loop. James Clerk Maxwell derived it again using hydrodynamics in his 1861 paper *On Physical Lines of Force* and it is now one of the Maxwell equations, which form the basis of classical electromagnetism.

3.1 Ampère's original circuital law

Ampère's law relates magnetic fields to electric currents that produce them. Ampère's law determines the magnetic field associated with a given current, or the current associated with a given magnetic field, provided that the electric field does not change over time. In its original form, Ampère's circuital law relates a magnetic field to its electric current source. The law can be written in two forms, the "integral form" and the "differential form". The forms are equivalent, and related by the Kelvin–Stokes theorem. It can also be written in terms of either the **B** or **H** magnetic fields. Again, the two forms are equivalent (see the "proof" section below).

Ampère's circuital law is now known to be a correct law of physics in a magnetostatic situation: The system is static except possibly for continuous steady currents within closed loops. In all other cases the law is incorrect unless Maxwell's correction is included (see below).

3.1.1 Integral form

In SI units (cgs units are later), the "integral form" of the original Ampère's circuital law is a line integral of the magnetic field around some closed curve C (arbitrary but must be closed). The curve C in turn bounds both a surface S which the electric current passes through (again arbitrary but not closed—since no three-dimensional volume is enclosed by S), and encloses the current. The mathematical statement of the law is a relation between the total amount of magnetic field around some path (line integral) due to the current which passes through that enclosed path (surface integral). It can be written in a number of forms.[2][3]

In terms of **total** current, which includes both **free** and **bound** current, the line integral of the magnetic B-field (in tesla, T) around closed curve C is proportional to the total current I_{enc} passing through a surface S (enclosed by C):

$$\oint_C \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \mu_0 I_{\text{enc}}$$

where $\mathbf{J}$ is the total current density (in ampere per square metre, Am^{-2}).

Alternatively in terms of **free** current, the line integral of the magnetic H-field (in ampere per metre, Am^{-1}) around closed curve C equals the free current $I_{\text{f, enc}}$ through a surface S:

$$\oint_C \mathbf{H} \cdot \mathrm{d}\boldsymbol{\ell} = \iint_S \mathbf{J}_{\text{f}} \cdot \mathrm{d}\mathbf{S} = I_{\text{f,enc}}$$

where $\mathbf{J}_{\text{f}}$ is the free current density only. Furthermore

- $\oint_C$ is the closed line integral around the closed curve C,
- $\iint_S$ denotes a 2d surface integral over S enclosed by C
- • is the vector dot product,
- $\mathrm{d}\boldsymbol{\ell}$ is an infinitesimal element (a differential) of the curve C (i.e. a vector with magnitude equal to the length of the infinitesimal line element, and direction given by the tangent to the curve C)
- $\mathrm{d}\mathbf{S}$ is the vector area of an infinitesimal element of surface S (that is, a vector with magnitude equal to the area of the infinitesimal surface element, and direction normal to surface S. The direction of the normal must correspond with the orientation of C by the right hand rule), see below for further explanation of the curve C and surface S.

The **B** and **H** fields are related by the constitutive equation

$$\mathbf{B} = \mu_0 \mathbf{H}$$

where μ_0 is the magnetic constant.

There are a number of ambiguities in the above definitions that require clarification and a choice of convention.

1. First, three of these terms are associated with sign ambiguities: the line integral $\oint_C$ could go around the loop in either direction (clockwise or counterclockwise); the vector area d**S** could point in either of the two directions normal to the surface; and I_{enc} is the net current passing through the surface S, meaning the current passing through in one direction, minus the current in the other direction—but either direction could be chosen as positive. These ambiguities are resolved by the right-hand rule: With the palm of the right-hand toward the area of integration, and the index-finger pointing along the direction of line-integration, the outstretched thumb points in the direction that must be chosen for the vector area d**S**. Also the current passing in the same direction as d**S** must be counted as positive. The right hand grip rule can also be used to determine the signs.

2. Second, there are infinitely many possible surfaces S that have the curve C as their border. (Imagine a soap film on a wire loop, which can be deformed by moving the wire). Which of those surfaces is to be chosen? If the loop does not lie in a single plane, for example, there is no one obvious choice. The answer is that it does not matter; it can be proven that any surface with boundary C can be chosen.

3.1.2 Differential form

By the Stokes' theorem, this equation can also be written in a "differential form". Again, this equation only applies in the case where the electric field is constant in time, meaning the currents are steady (time-independent, else the magnetic field would change with time); see below for the more general form. In SI units, the equation states for total current:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

and for free current

$$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f}$$

where **∇**× is the curl operator.

3.2 Note on free current versus bound current

The electric current that arises in the simplest textbook situations would be classified as "free current"—for example, the current that passes through a wire or battery. In contrast, "bound current" arises in the context of bulk materials that can be magnetized and/or polarized. (All materials can to some extent.)

When a material is magnetized (for example, by placing it in an external magnetic field), the electrons remain bound to their respective atoms, but behave as if they were orbiting the nucleus in a particular direction, creating a microscopic current. When the currents from all these atoms are put together, they create the same effect as a macroscopic current, circulating perpetually around the magnetized object. This magnetization current **J**M is one contribution to "bound current".

The other source of bound current is bound charge. When an electric field is applied, the positive and negative bound charges can separate over atomic distances in polarizable materials, and when the bound charges move, the polarization changes, creating another contribution to the "bound current", the polarization current **J**P.

The total current density **J** due to free and bound charges is then:

$$\mathbf{J} = \mathbf{J}_\mathrm{f} + \mathbf{J}_\mathrm{M} + \mathbf{J}_\mathrm{P}$$

with $\mathbf{J}_\mathrm{f}$ the "free" or "conduction" current density.

All current is fundamentally the same, microscopically. Nevertheless, there are often practical reasons for wanting to treat bound current differently from free current. For example, the bound current usually originates over atomic dimensions, and one may wish to take advantage of a simpler theory intended for larger dimensions. The result is that the more microscopic Ampère's law, expressed in terms of **B** and the microscopic current (which includes free, magnetization and polarization currents), is sometimes put into the equivalent form below in terms of **H** and the free current only. For a detailed definition of free current and bound current, and the proof that the two formulations are equivalent, see the "proof" section below.

3.3 Shortcomings of the original formulation of Ampère's circuital law

There are two important issues regarding Ampère's law that require closer scrutiny. First, there is an issue regarding the continuity equation for electrical charge. In vector calculus, the identity for the divergence of a curl states that a vector field's curl divergence must always be zero. Hence

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0$$

and so the original Ampère's law implies that

$$\nabla \cdot \mathbf{J} = 0.$$

But in general

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

which is non-zero for a time-varying charge density. An example occurs in a capacitor circuit where time-varying charge densities exist on the plates.[4][5][6][7][8]

Second, there is an issue regarding the propagation of electromagnetic waves. For example, in free space, where

$$\mathbf{J} = \mathbf{0},$$

Ampère's law implies that

$$\nabla \times \mathbf{B} = \mathbf{0}$$

but instead

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}.$$

To treat these situations, the contribution of displacement current must be added to the current term in Ampère's law.

James Clerk Maxwell conceived of displacement current as a polarization current in the dielectric vortex sea, which he used to model the magnetic field hydrodynamically and mechanically.[9] He added this displacement current to Ampère's circuital law at equation (112) in his 1861 paper *On Physical Lines of Force*.[10]

3.3.1 Displacement current

Main article: Displacement current

In free space, the displacement current is related to the time rate of change of electric field.

In a dielectric the above contribution to displacement current is present too, but a major contribution to the displacement current is related to the polarization of the individual molecules of the dielectric material. Even though charges cannot flow freely in a dielectric, the charges in molecules can move a little under the influence of an electric field. The positive and negative charges in molecules separate under the applied field, causing an increase in the state of polarization, expressed as the polarization density **P**. A changing state of polarization is equivalent to a current.

Both contributions to the displacement current are combined by defining the displacement current as:[4]

$$\mathbf{J}_\mathrm{D} = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},\, t)\ ,$$

where the electric displacement field is defined as:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \varepsilon_\mathrm{r} \mathbf{E}\ ,$$

where ε_0 is the electric constant, ε_r the relative static permittivity, and **P** is the polarization density. Substituting this form for **D** in the expression for displacement current, it has two components:

$$\mathbf{J}_\mathrm{D} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t}.$$

The first term on the right hand side is present everywhere, even in a vacuum. It doesn't involve any actual movement of charge, but it nevertheless has an associated magnetic field, as if it were an actual current. Some authors apply the name *displacement current* to only this contribution.[11]

The second term on the right hand side is the displacement current as originally conceived by Maxwell, associated with the polarization of the individual molecules of the dielectric material.

Maxwell's original explanation for displacement current focused upon the situation that occurs in dielectric media. In the modern post-aether era, the concept has been extended to apply to situations with no material media present, for example, to the vacuum between the plates of a charging vacuum capacitor. The displacement current is justified today because it serves several requirements of an electromagnetic theory: correct prediction of magnetic fields in regions where no free current flows; prediction of wave propagation of electromagnetic fields; and conservation of electric charge in cases where charge density is time-varying. For greater discussion see Displacement current.

3.4 Extending the original law: the Maxwell–Ampère equation

Next Ampère's equation is extended by including the polarization current, thereby remedying the limited applicability of the original Ampère's circuital law.

Treating free charges separately from bound charges, Ampère's equation including Maxwell's correction in terms of the **H**-field is (the **H**-field is used because it includes the magnetization currents, so **J**M does not appear explicitly, see H-field and also Note):[12]

$$\oint_C \mathbf{H} \cdot \mathrm{d}\boldsymbol{\ell} = \iint_S \left(\mathbf{J}_\mathrm{f} + \frac{\partial}{\partial t}\mathbf{D} \right) \cdot \mathrm{d}\mathbf{S}$$

(integral form), where **H** is the magnetic H field (also called "auxiliary magnetic field", "magnetic field intensity", or just "magnetic field"), **D** is the electric displacement field, and $\mathbf{J}_\mathrm{f}$ is the enclosed conduction current or free current density. In differential form,

$$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f} + \frac{\partial}{\partial t}\mathbf{D} \,.$$

On the other hand, treating all charges on the same footing (disregarding whether they are bound or free charges), the generalized Ampère's equation, also called the Maxwell–Ampère equation, is in integral form (see the "proof" section below):

In differential form,

In both forms **J** includes magnetization current density[13] as well as conduction and polarization current densities. That is, the current density on the right side of the Ampère–Maxwell equation is:

$$\mathbf{J}_\mathrm{f} + \mathbf{J}_\mathrm{D} + \mathbf{J}_\mathrm{M} = \mathbf{J}_\mathrm{f} + \mathbf{J}_\mathrm{P} + \mathbf{J}_\mathrm{M} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \,,$$

where current density **J**D is the *displacement current*, and **J** is the current density contribution actually due to movement of charges, both free and bound. Because $\nabla \cdot \mathbf{D} = \varrho$, the charge continuity issue with Ampère's original formulation is no longer a problem.[14] Because of the term in $\varepsilon_0 \partial \mathbf{E} / \partial t$, wave propagation in free space now is possible.

With the addition of the displacement current, Maxwell was able to hypothesize (correctly) that light was a form of electromagnetic wave. See electromagnetic wave equation for a discussion of this important discovery.

3.4.1 Proof of equivalence

3.5 Ampère's law in cgs units

In cgs units, the integral form of the equation, including Maxwell's correction, reads

$$\oint_C \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{1}{c} \iint_S \left(4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \mathrm{d}\mathbf{S}$$

where *c* is the speed of light.

The differential form of the equation (again, including Maxwell's correction) is

$$\nabla \times \mathbf{B} = \frac{1}{c} \left(4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \right).$$

3.6 See also

3.7 Notes

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[8] J.C. Slater and N.H. Frank (1969). *Electromagnetism* (Reprint of 1947 ed.). Courier Dover Publications. p. 83. ISBN 0-486-62263-0.

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[10] James C. Maxwell (1861). "On Physical Lines of Force" (PDF). *Philosophical Magazine and Journal of Science*.

[11] For example, see David J. Griffiths (1999). op. cit.. Upper Saddle River, NJ: Prentice Hall. p. 323. ISBN 0-13-805326-X. and Tai L. Chow (2006). *Introduction to Electromagnetic Theory*. Jones & Bartlett. p. 204. ISBN 0-7637-3827-1.

[12] Mircea S. Rogalski, Stuart B. Palmer (2006). *Advanced University Physics*. CRC Press. p. 267. ISBN 1-58488-511-4.

[13] Stuart B. Palmer & Mircea S. Rogalski (2006). *Advanced University Physics*. CRC Press. p. 251. ISBN 1-58488-511-4.

[14] The magnetization current can be expressed as the *curl* of the magnetization, so its divergence is zero and it does not contribute to the continuity equation. See magnetization current.

3.8 Further reading

- Griffiths, David J. (1998). *Introduction to Electrodynamics (3rd ed.)*. Prentice Hall. ISBN 0-13-805326-X.
- Tipler, Paul (2004). *Physics for Scientists and Engineers: Electricity, Magnetism, Light, and Elementary Modern Physics (5th ed.)*. W. H. Freeman. ISBN 0-7167-0810-8.

3.9 External links

- *Simple Nature* by Benjamin Crowell Ampere's law from an online textbook
- MISN-0-138 *Ampere's Law* (PDF file) by Kirby Morgan for Project PHYSNET.
- MISN-0-145 *The Ampere–Maxwell Equation; Displacement Current* (PDF file) by J.S. Kovacs for Project PHYSNET.
- *The Ampère's Law Song* (PDF file) by Walter Fox Smith; Main page, with recordings of the song.
- *A Dynamical Theory of the Electromagnetic Field* Maxwell's paper of 1864

Chapter 4

Divergence theorem

"Gauss's theorem" redirects here. For Gauss's theorem concerning the electric field, see Gauss's law.
"Ostrogradsky theorem" redirects here. For Ostrogradsky's theorem concerning the linear instability of the Hamiltonian associated with a Lagrangian dependent on higher time derivatives than the first, see Ostrogradsky instability.

In vector calculus, the **divergence theorem**, also known as **Gauss's theorem** or **Ostrogradsky's theorem**,[1][2] is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface.

More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Intuitively, it states that *the sum of all sources minus the sum of all sinks gives the net flow out of a region*.

The divergence theorem is an important result for the mathematics of engineering, in particular in electrostatics and fluid dynamics.

In physics and engineering, the divergence theorem is usually applied in three dimensions. However, it generalizes to any number of dimensions. In one dimension, it is equivalent to the fundamental theorem of calculus. In two dimensions, it is equivalent to Green's theorem.

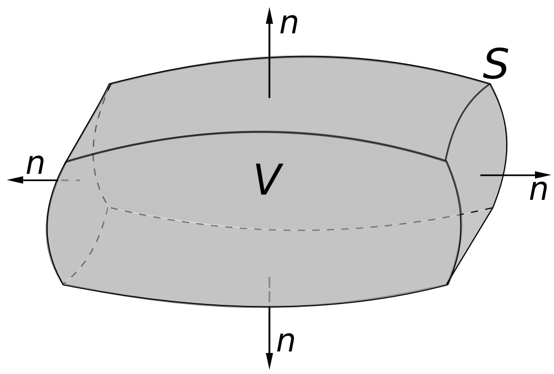
The theorem is a special case of the more general Stokes' theorem.[3]

4.1 Intuition

If a fluid is flowing in some area, then the rate at which fluid flows out of a certain region within that area can be calculated by adding up the sources inside the region and subtracting the sinks. The fluid flow is represented by a vector field, and the vector field's divergence at a given point describes the strength of the source or sink there. So, integrating the field's divergence over the interior of the region should equal the integral of the vector field over the region's boundary. The divergence theorem says that this is true.[4]

The divergence theorem is employed in any conservation law which states that the volume total of all sinks and sources, that is the volume integral of the divergence, is equal to the net flow across the volume's boundary.[5]

4.2 Mathematical statement

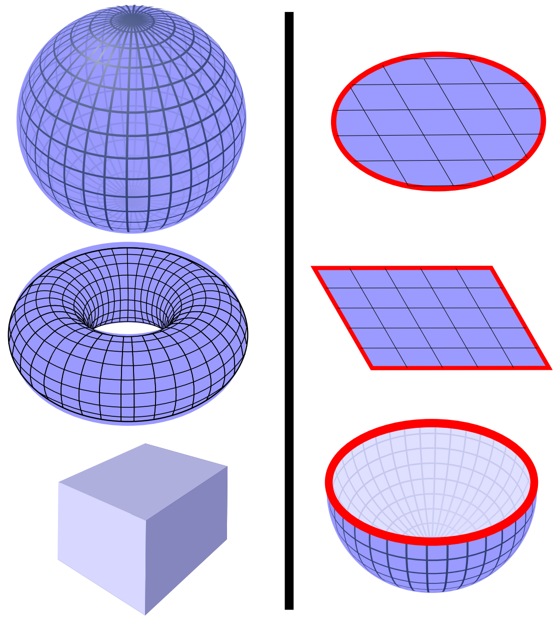


A region V bounded by the surface S = ∂V *with the surface normal n*

Suppose V is a subset of $\mathbb{R}^n$ (in the case of $n = 3$, V represents a volume in 3D space) which is compact and has a piecewise smooth boundary S (also indicated with $\partial V = S$). If $\mathbf{F}$ is a continuously differentiable vector field defined on a neighborhood of V, then we have:[6]

$$\iiint_V (\nabla\cdot\mathbf{F})\ dV = \oiint_S (\mathbf{F}\cdot\mathbf{n})\, dS.$$

The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V. The closed manifold ∂V is quite generally the boundary of V oriented by outward-pointing normals, and $\mathbf{n}$ is the outward pointing unit normal field of the boundary ∂V. ($d\mathbf{S}$ may be used as a shorthand for $\mathbf{n}dS$.) The symbol within the two integrals stresses once



The divergence theorem can be used to calculate a flux through a closed surface that fully encloses a volume, like any of the surfaces on the left. It can not directly be used to calculate the flux through surfaces with boundaries, like those on the right. (Surfaces are blue, boundaries are red.)

more that ∂V is a *closed* surface. In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume V, and the right-hand side represents the total flow across the boundary S.

4.2.1 Corollaries

By applying the divergence theorem in various contexts, other useful identities can be derived (cf. vector identities).[6]

- Applying the divergence theorem to the product of a scalar function g and a vector field **F**, the result is

$$\iiint_V [\mathbf{F}\cdot(\nabla g) + g\,(\nabla\cdot\mathbf{F})]\,dV = \oiint_S g\mathbf{F}\cdot d\mathbf{S}.$$

  A special case of this is $\mathbf{F} = \nabla f$, in which case the theorem is the basis for Green's identities.

- Applying the divergence theorem to the cross-product of two vector fields $\mathbf{F}\times\mathbf{G}$, the result is

$$\iiint_V [\mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})]\,dV = \oiint_S (\mathbf{F}\times\mathbf{G})\cdot d\mathbf{S}.$$

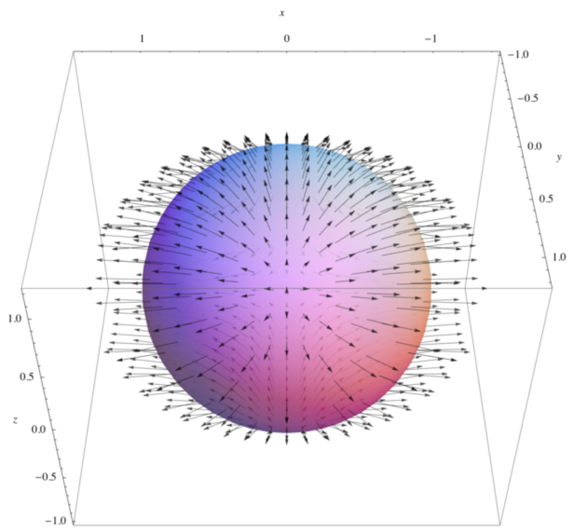
- Applying the divergence theorem to the product of a scalar function, f, and a non-zero constant vector **c**, the following theorem can be proven:[7]

$$\iiint_V \mathbf{c}\cdot\nabla f\,dV = \oiint_S (\mathbf{c}f)\cdot d\mathbf{S} - \iiint_V f(\nabla\cdot\mathbf{c})\,dV.$$

- Applying the divergence theorem to the cross-product of a vector field **F** and a non-zero constant vector **c**, the following theorem can be proven:[7]

$$\iiint_V \mathbf{c}\cdot(\nabla\times\mathbf{F})\,dV = \oiint_S (\mathbf{F}\times\mathbf{c})\cdot d\mathbf{S}.$$

4.3 Example



The vector field corresponding to the example shown. Note, vectors may point into or out of the sphere.

Suppose we wish to evaluate

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

where S is the unit sphere defined by

$$S = \left\{x, y, z \in \mathbb{R}^3 \ :\ x^2 + y^2 + z^2 = 1\right\}.$$

and **F** is the vector field

$$\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

The direct computation of this integral is quite difficult, but we can simplify the derivation of the result using the divergence theorem, because the divergence theorem says that the integral is equal to:

$$\iiint_W (\nabla\cdot\mathbf{F})\,dV = 2\iiint_W (1+y+z)\,dV = 2\iiint_W dV + 2\iiint_W y\,dV + 2\iiint_W z\,dV.$$

where W is the unit ball:

$$W = \left\{x, y, z \in \mathbb{R}^3\ :\ x^2 + y^2 + z^2 \le 1\right\}.$$

Since the function y is positive in one hemisphere of W and negative in the other, in an equal and opposite way, its total integral over W is zero. The same is true for z:

$$\iiint_W y\,dV = \iiint_W z\,dV = 0.$$

Therefore,

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,dS = 2\iiint_W dV = \tfrac{8\pi}{3},$$

because the unit ball W has volume $4\pi/3$.

4.4 Applications

4.4.1 *Differential form* and *integral form* of physical laws

As a result of the divergence theorem, a host of physical laws can be written in both a differential form (where one quantity is the divergence of another) and an integral form (where the flux of one quantity through a closed surface is equal to another quantity). Three examples are Gauss's law (in electrostatics), Gauss's law for magnetism, and Gauss's law for gravity.

Continuity equations

Main article: continuity equation

Continuity equations offer more examples of laws with both differential and integral forms, related to each other by the divergence theorem. In fluid dynamics, electromagnetism, quantum mechanics, relativity theory, and a number of other fields, there are continuity equations that describe the conservation of mass, momentum, energy, probability, or other quantities. Generically, these equations state that the divergence of the flow of the conserved quantity is equal to the distribution of *sources* or *sinks* of that quantity. The divergence theorem states that any such continuity equation can be written in a differential form (in terms of a divergence) and an integral form (in terms of a flux).[8]

4.4.2 Inverse-square laws

Any *inverse-square law* can instead be written in a *Gauss' law*-type form (with a differential and integral form, as described above). Two examples are Gauss' law (in electrostatics), which follows from the inverse-square Coulomb's law, and Gauss' law for gravity, which follows from the inverse-square Newton's law of universal gravitation. The derivation of the Gauss' law-type equation from the inverse-square formulation (or vice versa) is exactly the same in both cases; see either of those articles for details.[8]

4.5 History

The theorem was first discovered by Lagrange in 1762,[9] then later independently rediscovered by Gauss in 1813,[10] by Ostrogradsky, who also gave the first proof of the general theorem, in 1826,[11] by Green in 1828,[12] etc.[13] Subsequently, variations on the divergence theorem are correctly called Ostrogradsky's theorem, but also commonly Gauss's theorem, or Green's theorem.

4.6 Examples

To verify the planar variant of the divergence theorem for a region R:

$$R = \left\{x, y \in \mathbb{R}^2\ :\ x^2 + y^2 \le 1\right\},$$

and the vector field:

$$\mathbf{F}(x, y) = 2y\mathbf{i} + 5x\mathbf{j}.$$

The boundary of R is the unit circle, C, that can be represented parametrically by:

$$x = \cos(s), \quad y = \sin(s)$$

such that $0 \le s \le 2\pi$ where s units is the length arc from the point $s = 0$ to the point P on C. Then a vector equation of C is

$$C(s) = \cos(s)\mathbf{i} + \sin(s)\mathbf{j}.$$

At a point P on C:

$$P = (\cos(s), \sin(s)) \Rightarrow \mathbf{F} = 2\sin(s)\mathbf{i} + 5\cos(s)\mathbf{j}.$$

Therefore,

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_0^{2\pi} (2\sin(s)\mathbf{i} + 5\cos(s)\mathbf{j})\cdot(\cos(s)\mathbf{i} + \sin(s)\mathbf{j})\,ds \\ &= \int_0^{2\pi} (2\sin(s)\cos(s) + 5\sin(s)\cos(s))\,ds \\ &= 7\int_0^{2\pi} \sin(s)\cos(s)\,ds \\ &= 0.\end{aligned}$$

Because $M = 2y$, $\partial M/\partial x = 0$, and because $N = 5x$, $\partial N/\partial y = 0$. Thus

$$\iint_R \operatorname{div}\mathbf{F}\,dA = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = 0.$$

4.7 Generalizations

4.7.1 Multiple dimensions

One can use the general Stokes' Theorem to equate the n-dimensional volume integral of the divergence of a vector field **F** over a region U to the (n – 1)-dimensional surface integral of **F** over the boundary of U:

$$\int_U \nabla\cdot\mathbf{F}\,dV_n = \oint_{\partial U} \mathbf{F}\cdot\mathbf{n}\,dS_{n-1}$$

This equation is also known as the Divergence theorem.

When n = 2, this is equivalent to Green's theorem.

When n = 1, it reduces to the Fundamental theorem of calculus.

4.7.2 Tensor fields

Main article: Tensor field

Writing the theorem in Einstein notation:

$$\iiint_V \frac{\partial \mathbf{F}_i}{\partial x_i} dV = \oiint_S \mathbf{F}_i n_i\,dS$$

suggestively, replacing the vector field **F** with a rank-n tensor field T, this can be generalized to:[14]

$$\iiint_V \frac{\partial T_{i_1 i_2\cdots i_q\cdots i_n}}{\partial x_{i_q}} dV = \oiint_S T_{i_1 i_2\cdots i_q\cdots i_n} n_{i_q}\,dS.$$

where on each side, tensor contraction occurs for at least one index. This form of the theorem is still in 3d, each index takes values 1, 2, and 3. It can be generalized further still to higher (or lower) dimensions (for example to 4d spacetime in general relativity[15]).

4.8 See also

- Stokes' theorem
- Kelvin–Stokes theorem

4.9 Notes

[1] or less correctly as **Gauss' theorem** (see history for reason)

[2] Katz, Victor J. (1979). "The history of Stokes's theorem". *Mathematics Magazine* (Mathematical Association of America) **52**: 146–156. doi:10.2307/2690275. reprinted in Anderson, Marlow (2009). *Who Gave You the Epsilon?: And Other Tales of Mathematical History*. Mathematical Association of America. pp. 78–79. ISBN 0883855690.

[3] Stewart, James (2008), "Vector Calculus", *Calculus: Early Transcendentals* (6 ed.), Thomson Brooks/Cole, ISBN 978-0-495-01166-8

[4] R. G. Lerner, G. L. Trigg (1994). *Encyclopaedia of Physics* (2nd ed.). VHC. ISBN 3-527-26954-1.

[5] Byron, Frederick; Fuller, Robert (1992), *Mathematics of Classical and Quantum Physics*, Dover Publications, p. 22, ISBN 978-0-486-67164-2

[6] M. R. Spiegel; S. Lipschutz; D. Spellman (2009). *Vector Analysis*. Schaum's Outlines (2nd ed.). USA: McGraw Hill. ISBN 978-0-07-161545-7.

[7] MathWorld

[8] C.B. Parker (1994). *McGraw Hill Encyclopaedia of Physics* (2nd ed.). McGraw Hill. ISBN 0-07-051400-3.

[9] In his 1762 paper on sound, Lagrange treats a special case of the divergence theorem: Lagrange (1762) "Nouvelles recherches sur la nature et la propagation du son" (New researches on the nature and propagation of sound), *Miscellanea Taurinensia* (also known as: *Mélanges de Turin*), **2**: 11 - 172. This article is reprinted as: "Nouvelles recherches sur la nature et la propagation du son" in: J.A. Serret, ed., *Oeuvres de Lagrange*, (Paris, France: Gauthier-Villars, 1867), vol. 1, pages 151-316; on pages 263-265, Lagrange transforms triple integrals into double integrals using integration by parts.

[10] C. F. Gauss (1813) "Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata," *Commentationes societatis regiae scientiarium Gottingensis recentiores*, **2**: 355-378; Gauss considered a special case of the theorem; see the 4th, 5th, and 6th pages of his article.

[11] Mikhail Ostragradsky presented his proof of the divergence theorem to the Paris Academy in 1826; however, his work was not published by the Academy. He returned to St. Petersburg, Russia, where in 1828-1829 he read the work that he'd done in France, to the St. Petersburg Academy, which published his work in abbreviated form in 1831.

- His proof of the divergence theorem -- "Démonstration d'un théorème du calcul intégral" (Proof of a theorem in integral calculus) -- which he had read to the Paris Academy on February 13, 1826, was translated, in 1965, into Russian together with another article by him. See: Юшкевич А.П. (Yushkevich A.P.) and Антропова В.И. (Antropov V.I.) (1965) "Неопубликованные работы М.В. Остроградского" (Unpublished works of MV Ostrogradskii), *Историко-математические исследования* (Istoriko-Matematicheskie Issledovaniya / Historical-Mathematical Studies), **16**: 49-96; see the section titled: "Остроградский М.В. Доказательство одной теоремы интегрального исчисления" (Ostrogradskii M. V. Dokazatelstvo odnoy teoremy integralnogo ischislenia / Ostragradsky M.V. Proof of a theorem in integral calculus).
- M. Ostrogradsky (presented: November 5, 1828 ; published: 1831) "Première note sur la théorie de la chaleur" (First note on the theory of heat) *Mémoires de l'Académie impériale des sciences de St. Pétersbourg*, series 6, **1**: 129-133; for an abbreviated version of his proof of the divergence theorem, see pages 130-131.
- Victor J. Katz (May1979) "The history of Stokes' theorem," *Mathematics Magazine*, **52**(3): 146-156; for Ostragradsky's proof of the divergence theorem, see pages 147-148.

[12] George Green, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* (Nottingham, England: T. Wheelhouse, 1838). A form of the "divergence theorem" appears on pages 10-12.

[13] Other early investigators who used some form of the divergence theorem include:

- Poisson (presented: February 2, 1824 ; published: 1826) "Mémoire sur la théorie du magnétisme" (Memoir on the theory of magnetism), *Mémoires de l'Académie des sciences de l'Institut de France*, **5**: 247-338; on pages 294-296, Poisson transforms a volume integral (which is used to evaluate a quantity Q) into a surface integral. To make this transformation, Poisson follows the same procedure that is used to prove the divergence theorem.
- Frédéric Sarrus (1828) "Mémoire sur les oscillations des corps flottans" (Memoir on the oscillations of floating bodies), *Annales de mathématiques pures et appliquées* (Nismes), **19**: 185-211.

[14] K.F. Riley, M.P. Hobson, S.J. Bence (2010). *Mathematical methods for physics and engineering*. Cambridge University Press. ISBN 978-0-521-86153-3.

[15] see for example:
J.A. Wheeler, C. Misner, K.S. Thorne (1973). *Gravitation*. W.H. Freeman & Co. pp. 85–86, §3.5. ISBN 0-7167-0344-0., and
R. Penrose (2007). *The Road to Reality*. Vintage books. ISBN 0-679-77631-1.

4.10 External links

- Hazewinkel, Michiel, ed. (2001), "Ostrogradski formula", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Differential Operators and the Divergence Theorem at MathPages
- The Divergence (Gauss) Theorem by Nick Bykov, Wolfram Demonstrations Project.
- Weisstein, Eric W., "Divergence Theorem", *MathWorld*. *— This article was originally based on the GFDL article from PlanetMath at http://planetmath.org/encyclopedia/Divergence.html*

Chapter 5

Electric displacement field

In physics, the **electric displacement field**, denoted by **D**, is a vector field that appears in Maxwell's equations. It accounts for the effects of free and bound charge within materials. "**D**" stands for "displacement", as in the related concept of displacement current in dielectrics. In free space, the electric displacement field is equivalent to flux density, a concept that lends understanding to Gauss's law. It has the SI units of coulomb per squared metre (C m^{-2}).

5.1 Definition

In a dielectric material the presence of an electric field **E** causes the bound charges in the material (atomic nuclei and their electrons) to slightly separate, inducing a local electric dipole moment. The electric displacement field **D** is defined as

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P},$$

where ε_0 is the vacuum permittivity (also called permittivity of free space), and **P** is the (macroscopic) density of the permanent and induced electric dipole moments in the material, called the polarization density. Separating the total volume charge density into free and bound charges:

$$\rho = \rho_f + \rho_b$$

the density can be rewritten as a function of the polarization **P**:

$$\rho = \rho_f - \nabla \cdot \mathbf{P}.$$

The polarization **P** is defined to be a vector field whose divergence yields the density of bound charges ρ_b in the material. The electric field satisfies the equation:

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho = \frac{1}{\varepsilon_0}(\rho_f - \nabla \cdot \mathbf{P})$$

and hence

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

The displacement field therefore satisfies Gauss's law in a dielectric:

$$\nabla \cdot \mathbf{D} = \rho - \rho_b = \rho_f$$

However, electrostatic forces on ions or electrons in the material are still governed by the electric field **E** in the material via the Lorentz Force. It should also be remembered that **D** is not determined exclusively by the free charge. Consider the relationship:

$$\nabla \times \mathbf{D} = \varepsilon_0 \nabla \times \mathbf{E} + \nabla \times \mathbf{P},$$

which, by the fact that **E** has a curl of zero in electrostatic situations, evaluates to:

$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P}$$

The effect of this equation can be seen in the case of an object with a "frozen in" polarization like a bar electret, the electric analogue to a bar magnet. There is no free charge in such a material, but the inherent polarization gives rise to an electric field. If the wayward student were to assume that the **D** field were entirely determined by the free charge, he or she would conclude that the electric field were zero outside such a material, but this is patently not true. The electric field can be properly determined by using the above relation along with other boundary conditions on the polarization density to yield the bound charges, which will, in turn, yield the electric field.

In a linear, homogeneous, isotropic dielectric with instantaneous response to changes in the electric field, **P** depends linearly on the electric field,

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E},$$

where the constant of proportionality χ is called the electric susceptibility of the material. Thus

$$\mathbf{D} = \varepsilon_0(1+\chi)\mathbf{E} = \varepsilon\mathbf{E}$$

where $\varepsilon = \varepsilon_0\ \varepsilon_r$ is the permittivity, and $\varepsilon_r = 1 + \chi$ the relative permittivity of the material.

In linear, homogeneous, isotropic media, ε is a constant. However, in linear anisotropic media it is a tensor, and in nonhomogeneous media it is a function of position inside the medium. It may also depend upon the electric field (nonlinear materials) and have a time dependent response. Explicit time dependence can arise if the materials are physically moving or changing in time (e.g. reflections off a moving interface give rise to Doppler shifts). A different form of time dependence can arise in a time-invariant medium, in that there can be a time delay between the imposition of the electric field and the resulting polarization of the material. In this case, **P** is a convolution of the impulse response susceptibility χ and the electric field **E**. Such a convolution takes on a simpler form in the frequency domain—by Fourier transforming the relationship and applying the convolution theorem, one obtains the following relation for a linear time-invariant medium:

$$\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega),$$

where ω is the frequency of the applied field. The constraint of causality leads to the Kramers–Kronig relations, which place limitations upon the form of the frequency dependence. The phenomenon of a frequency-dependent permittivity is an example of material dispersion. In fact, all physical materials have some material dispersion because they cannot respond instantaneously to applied fields, but for many problems (those concerned with a narrow enough bandwidth) the frequency-dependence of ε can be neglected.

At a boundary, $(\mathbf{D_1} - \mathbf{D_2}) \cdot \hat{\mathbf{n}} = D_{1,\perp} - D_{2,\perp} = \sigma_f$, where σ_f is the free charge density and the unit normal $\hat{\mathbf{n}}$ points in the direction from medium 2 to medium 1.[1]

5.2 History

Recall that Gauss's law was formulated by Carl Friedrich Gauss in 1835, but was not published until 1867. This places a lower limit on the year of the formulation and use of **D** to not earlier than 1835. Again, considering the law in its usual form using **E** vector rather than **D** vector, it can be safely said that **D** formalism was not used earlier than 1860s.

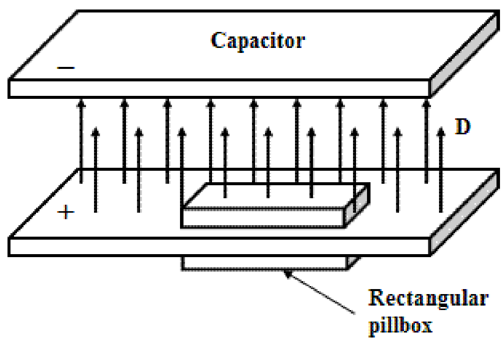
The Electric Displacement field was first found to be used in the year 1864 by James Clerk Maxwell, in his paper A Dynamical Theory of the Electromagnetic Field. Maxwell used calculus to exhibit Michael Faraday's theory, that light is an electromagnetic phenomenon. In PART V. — THEORY OF CONDENSERS, Maxwell introduced the term **D** in page 494, titled, Specific Capacity of Electric Induction (D), explained in a form different from the modern and familiar notations.

Confusion over the term "Maxwell's equations" arises because it has been used for a set of eight equations that appeared in Part III of Maxwell's 1864 paper A Dynamical Theory of the Electromagnetic Field, entitled "General Equations of the Electromagnetic Field", and this confusion is compounded by the writing of six of those eight equations as three separate equations (one for each of the Cartesian axes), resulting in twenty equations and twenty unknowns. (As noted above, this terminology is not common: Modern references to the term "Maxwell's equations" refer to the Heaviside restatements.)

It was Oliver Heaviside who reformulated the complicated Maxwell's equations to the modern, elegant form that we know today. But it wasn't until 1884 that Heaviside, concurrently with similar work by Willard Gibbs and Heinrich Hertz, grouped them together into a distinct set. This group of four equations was known variously as the Hertz–Heaviside equations and the Maxwell–Hertz equations, and are sometimes still known as the Maxwell–Heaviside equations.

Hence, it was probably Heaviside who lent **D** the present significance it now has.[2]

5.3 Example: Displacement field in a capacitor



A parallel plate capacitor. Using an imaginary pillbox, it is possible to use Gauss's law to explain the relationship between electric displacement and free charge.

Consider an infinite parallel plate capacitor placed in space (or in a medium) with no free charges present except on the capacitor. In SI units, the charge density on the plates is equal to the value of the **D** field between the plates. This follows directly from Gauss's law, by integrating over a small rectangular pillbox straddling one plate of the capacitor:

$$\oint_A \mathbf{D} \cdot \mathrm{d}\mathbf{A} = Q_{\text{free}}$$

On the sides of the pillbox, d$\mathbf{A}$ is perpendicular to the field, so that part of the integral is zero, leaving, for the space inside the capacitor where the fields of the two plates add,

$$|\mathbf{D}| = \frac{Q_{\text{free}}}{A}$$

where A is surface area of the top face of the small rectangular pillbox and Q_{free} / A is just the free surface charge density on the positive plate. Outside the capacitor, the fields of the two plates cancel each other and $|\mathbf{E}| = |\mathbf{D}| = 0$. If the space between the capacitor plates is filled with a linear homogeneous isotropic dielectric with permittivity ε, the total electric field $\mathbf{E}$ between the plates will be smaller than $\mathbf{D}$ by a factor of ε: $|\mathbf{E}| = Q_{\text{free}} / (\varepsilon A)$.

If the distance d between the plates of a *finite* parallel plate capacitor is much smaller than its lateral dimensions we can approximate it using the infinite case and obtain its capacitance as

$$C = \frac{Q_{\text{free}}}{V} \approx \frac{Q_{\text{free}}}{|\mathbf{E}|d} = \frac{A}{d}\varepsilon,$$

where V is the potential difference sustained between the two plates. The partial cancellation of fields in the dielectric allows a larger amount of free charge to dwell on the two plates of the capacitor per unit potential drop than would be possible if the plates were separated by vacuum.

5.4 See also

- History of Maxwell's equations#The term Maxwell's equations
- Polarization density
- Electric susceptibility
- Magnetizing field
- Electric dipole moment

5.5 References

[1] David Griffiths. *Introduction to Electrodynamics* (3rd 1999 ed.).

[2] History of Maxwell's equations#The term Maxwell's equations

- "electric displacement field" at PhysicsForums

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5.6.1 Text

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